Muon Particle Tracking

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Tuesday, February 5, 2002
Cooling Theory and Simulations Day
IIT

Outline

I. Boris Particle Push

- Spatial Boris Push Equations
- Stability and Accuracy
- Comparison with Runge-Kutta

II. Molière Scatter

- General Formalism
- Muon Kinematics
- Magnetic Field
- Error Estimates

Spatial Boris Push

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In collab. with P. Stoltz, J. Cary, at Tech-X (& U. Colorado, Boulder)

The spatial Boris scheme exchanges U for p_z and t for z

First, we replace the equation for pz with the equation for U:

$$\frac{dp_x}{dt} = q(E_x + v_y B_z - v_z B_y)$$
$$\frac{dp_y}{dt} = q(E_y + v_z B_x - v_x B_z)$$
$$\frac{d(U/c)}{dt} = q(E_x v_x + E_y v_y + E_z v_z)$$

Replacing t with z, the governing equations of the spatial Boris scheme are:

$$\frac{dp_x}{dz} = \frac{1}{v_z}\frac{dp_x}{dt} = q\left(\frac{E_x}{v_z} + \frac{v_y B_z}{v_z} - B_y\right)$$
$$\frac{dp_y}{dz} = \frac{1}{v_z}\frac{dp_y}{dt} = q\left(\frac{E_y}{v_z} - \frac{v_x B_z}{v_z} + B_y\right)$$
$$\frac{dU/c}{dz} = \frac{1}{v_z}\frac{dU/c}{dt} = q\left(\frac{v_x E_x}{v_z c} + \frac{v_y E_y}{v_z c} + \frac{E_z}{c}\right)$$

Decomposition (1)

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For spatial Boris push, the equations separate into terms that directly change p_z and terms that don't

- In the temporal Boris scheme, the separation is into one piece that changes U (E-fields) and one that doesn't (B-fields)

The terms that directly change p_z are E_z, B_x, and B_y

$$\frac{d}{dz}\begin{pmatrix} p_x \\ p_y \\ U/c \end{pmatrix} = \frac{q}{p_z}\begin{pmatrix} 0 & B_z & E_x/c \\ -B_z & 0 & E_y/c \\ E_x/c & E_y/c & 0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ U/c \end{pmatrix} + q\begin{pmatrix} -B_y \\ B_x \\ E_z/c \end{pmatrix}$$

For simplicity, rewrite: $$\frac{dw}{dz} = Mw + b$$

Decomposition (2)

The Boris scheme integrates vector and matrix terms separately

The Boris scheme says first push the vector term one-half step:

$$w^- = w^n + \frac{\Delta z}{2} b$$

Then push the matrix term a full step:

This step is implicit and requires some more massaging

$$w^+ - w^- = M \left(\frac{w^+ + w^-}{2} \right) \Delta z$$

Finally, push the vector term the final half step:

$$w^{n+1} = w^+ + \frac{\Delta z}{2} b$$

Explicit Expression for Boris Push

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Step-centered push of matrix term is 2nd-order accurate

Because M is constant, a step-centered scheme will be 2nd-order accurate

$$w^+ - w^- = M\left(\frac{w^+ + w^-}{2}\right)\Delta z$$

Solving for w^+ gives:

$$w^+ = \left(I - M\frac{\Delta z}{2}\right)^{-1}\left(I + M\frac{\Delta z}{2}\right) w^-$$

$$= (I+R)\, w^-,$$

$$R = \frac{q\Delta z/p_z}{1 + \frac{q^2\Delta z^2}{4p_z^2}\left(B_z^2 - \frac{E_x^2+E_y^2}{c^2}\right)}\begin{pmatrix} \frac{q\Delta z}{2p_z}\left(\frac{E_x^2}{c^2} - B_z^2\right) & B_z + \frac{q\Delta z}{2p_z}\frac{E_xE_y}{c^2} & \frac{E_x}{c} + \frac{q\Delta z}{2p_z}\frac{B_zE_y}{c} \\ -B_z + \frac{q\Delta z}{2p_z}\frac{E_xE_y}{c^2} & \frac{q\Delta z}{2p_z}\left(\frac{E_y^2}{c^2} - B_z^2\right) & \frac{E_y}{c} - \frac{q\Delta z}{2p_z}\frac{B_zE_x}{c} \\ \frac{E_x}{c} + \frac{q\Delta z}{2p_z}\frac{B_zE_y}{c} & \frac{E_y}{c} + \frac{q\Delta z}{2p_z}\frac{B_zE_x}{c} & \frac{q\Delta z}{2p_z}\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2}\right) \end{pmatrix}$$

Integration Cycle

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Start

Leap-frog: push the positions one-half step

Evaluate Fields

Boris: push the vector term one-half step

Boris: push the matrix term a full step

Boris: push the vector term the final half step

Leap-frog: push the positions one-half step

End

Boris Push Simplifies Tracking

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Spatial motion is calculated a 1/2 step off from momentum/energy evolution:

- typically, use leap-frog
- in ICOOL, split into two half-steps, before and after evolution of w.

Fields are only evaluated once, where momentum kick is applied

- compared to 4 field evaluations for RK
- local effect on particle, so almost indep. of coordinates
- track as if no field, then replace Δz with separation between planes (now a function of transverse co-ords)

As in RK, assumes small energy loss per step.

Boris Push is Space-Symmetric

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– except energy loss, scatter, are applied at end of step

Second-order conservation of energy

- also canonical momentum when applicable
- robust for large stepsizes

Errors tend to average out

- in RK scheme, errors will slowly accumulate

Both schemes work well for small phase advances, but Boris push is simpler to calculate

- especially if field calculations are expensive
- less savings for curvilinear (where even field-free is complicated)

FIGURES

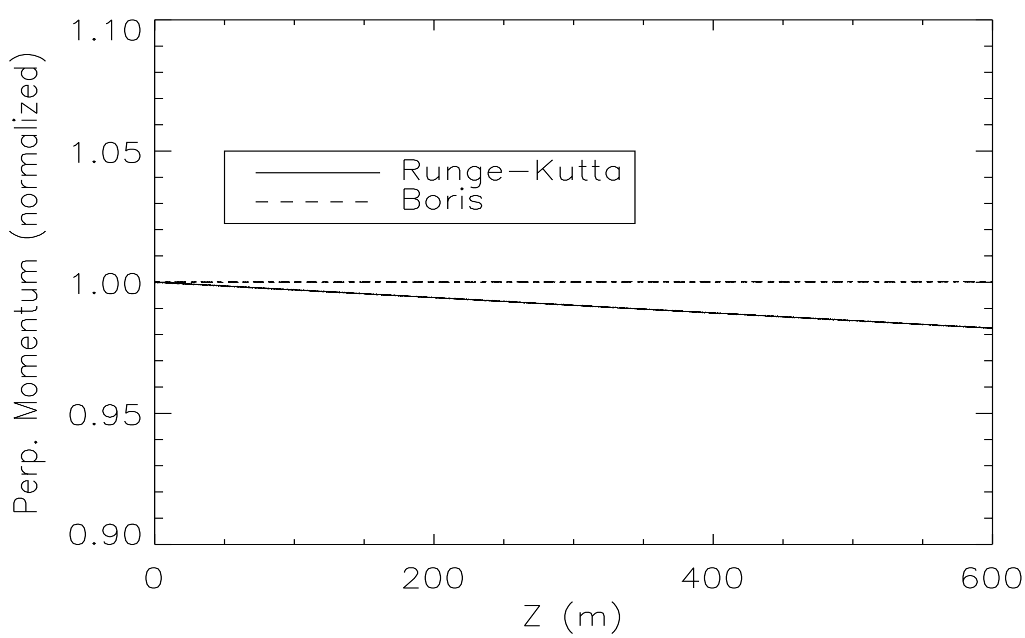

FIG. 1. The perpendicular momentum ($p_{perp} = \sqrt{p_x^2 + p_y^2}$), normalized to its initial value, as a function of distance along a uniform solenoid channel. The solid line is from a simulation using a Runge-Kutta integration scheme. The dashed line is from a simulation using the modified Boris integration scheme. The simulation using the modified Boris scheme conserved perpendicular momentum to better than 0.02%, more than two orders of magnitude better than the Runge-Kutta.

Boris Push Example

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Test limits of accuracy to see differences between RK and Boris.

Pass particle through single 2 m solenoid, with 7 T field.

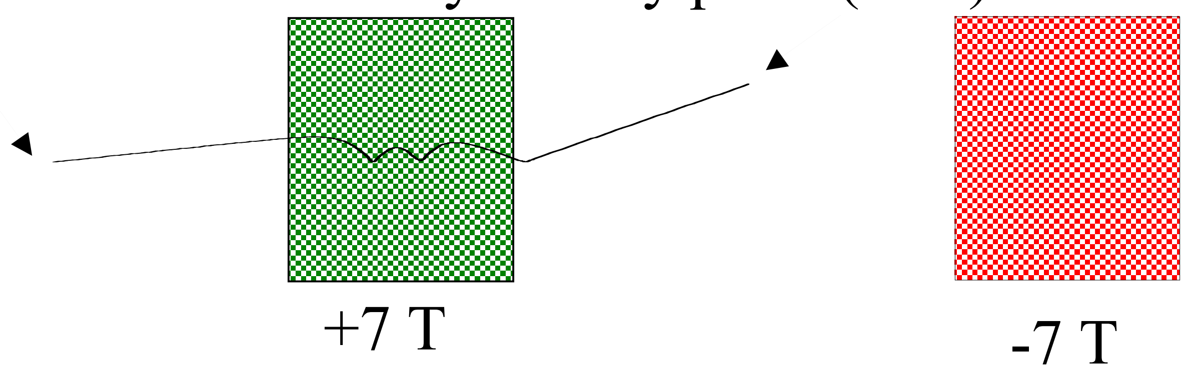
For 10 cm steps, noticeable discrepancies in angular momentum

(smaller errors: neglected r^3 term in vector potential)

Geometry: start off with r=0, P_Z = 200 MeV, P_X = 2 MeV

no vector potential at endpoints:

r = 0 symmetry point (B≡0)



Runge-Kutta results

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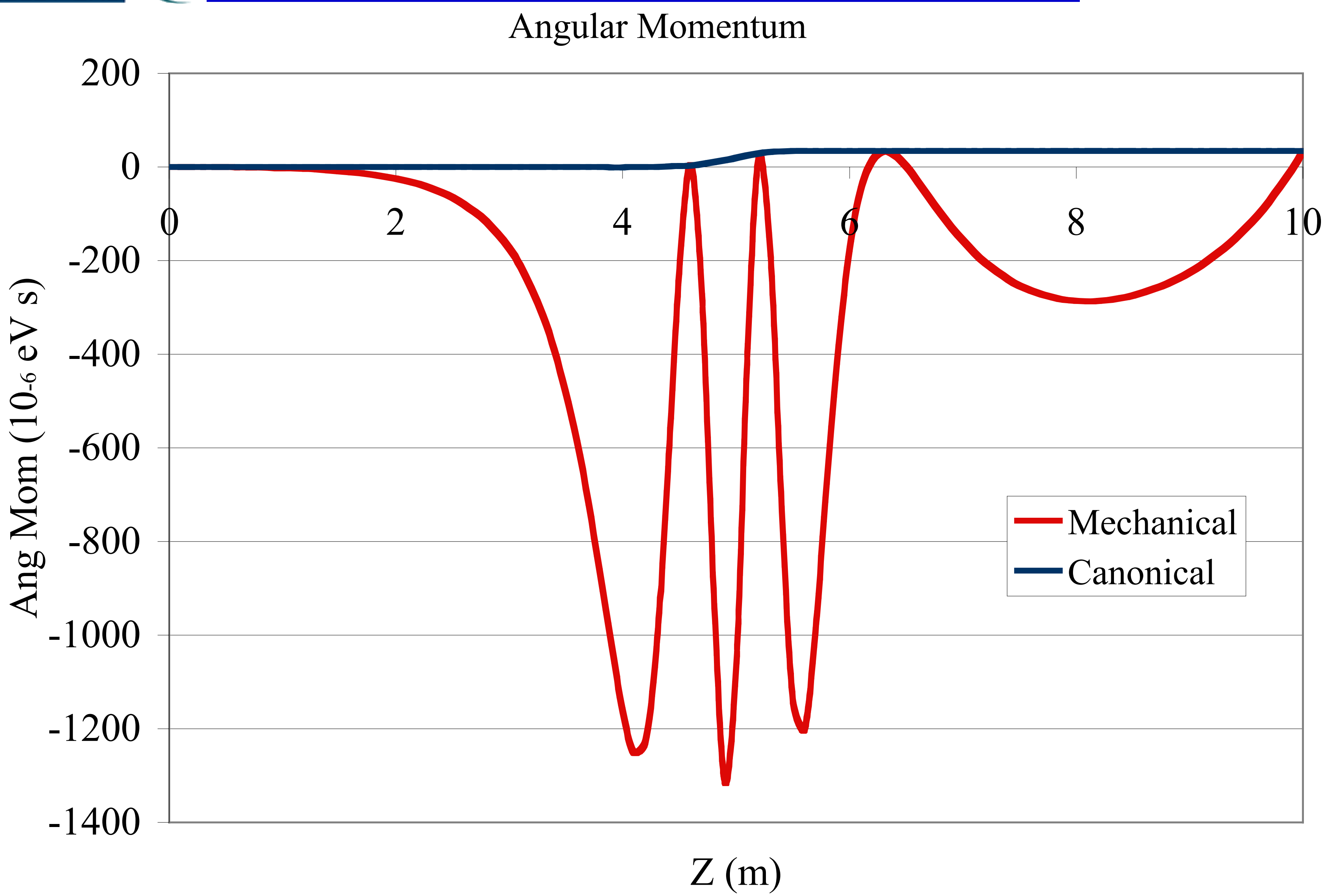

Boris-RK comparison

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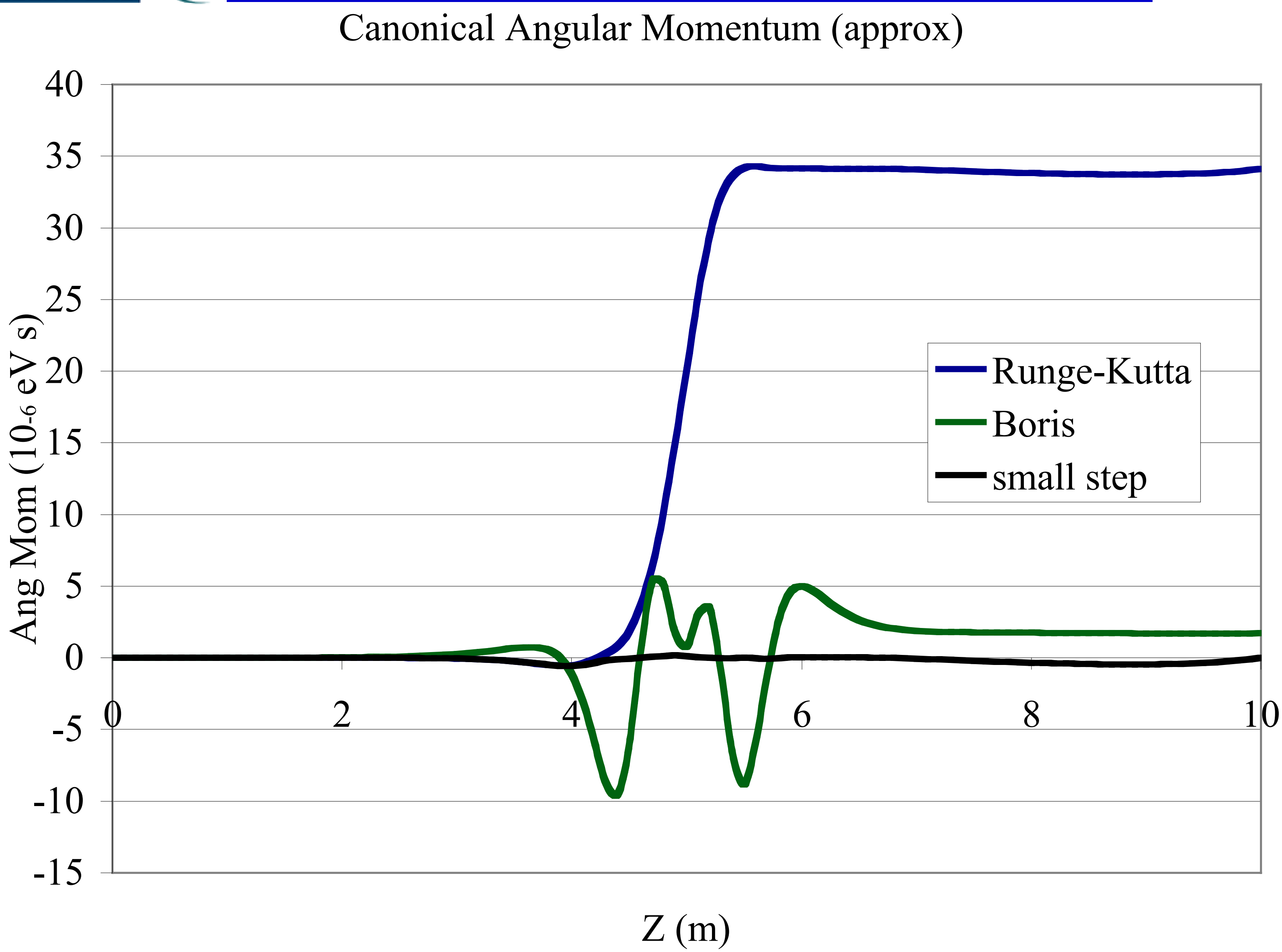

Moliere Scatter

Moliere Scatter Basics:

- constructs net scattering cross section out of many statistically independent scatter events
- Rutherford scatter (with electrons thrown in) parametrized as

  $N\, \Delta z\, s(\chi)\, \chi\, d\chi = 2\, \chi_C^2\, \chi\, d\chi\, q(\chi) / \chi_C^2$

  $\chi_C^2 = 4\pi\, N\, (\Delta z)\, e^4\, Z\, (Z+1) / (pv)^2$

  q() is screening, 1 for large χ, drops off to 0 at very small χ
- neglects energy loss between scatters, and effect on position
- result is Gaussian with enhanced tail
- total momentum scattering expressed in terms of single parameter:

  screening angle χ_α'
- requires minimum # of scatters ($\Omega_0 > 20$)
- large steps, limited by energy loss, cannot use PDG upper limits.

New Factors to Consider

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relatively large muon mass:

- keep m_μ/m_p corrections
- limits angle of scatter off of electrons; sharp cutoff
- also, Bielajew et al., partial wave method -- more accurate way to treat electron screening, mostly important for high Z

B field:

- no effect on angles; what about emittance growth?

finite step size:

- spatial motion as well as change in angle (result of many collisions)
- suffices to have step size $< \beta_\perp$

muons lose more energy for given scatter:

- good for cooling
- more constraints on step size (cannot be too large)

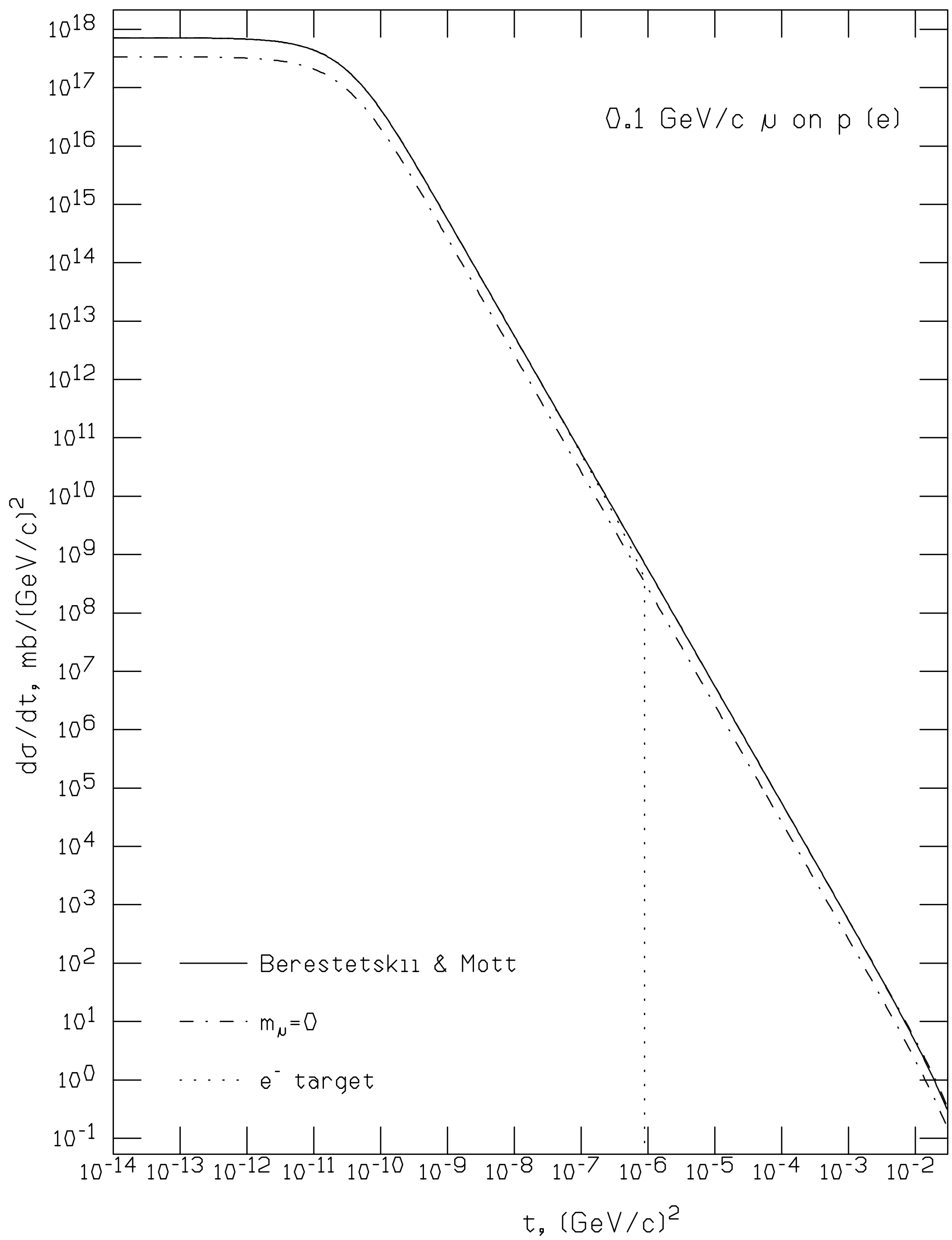

Figure 1: Differential cross section for μp scattering for muon momentum of $0.1\,GeV/c$, as predicted by Eq. 1 (Berestetskii), Eq. 2 ($m_\mu = 0$), Eq. 3 (Mott). Prediction of Eq. 4 for μe scattering is also shown. In all cases an atomic form factor (Eq. 5) is applied. Berestetskii and Mott predictions differ by about 10% at the kinematic limit but appear indistinguishable due to the very compressed ordinate scale.

Spatial Displacements due to Scatter

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We are simulating the result of many scatter events, at different points along orbit:

- yields correlated shifts in position
- and spatial diffusion

These spatial effects are currently neglected: they are modified by magnetic fields, but this is the only effect of uniform B

- what about rapidly varying B field?

B field reduces diffusion: B=0 is then "worst" for accuracy

Can put limits on errors from applying only momentum scatter, and only at end of step.

Scatter in Magnetic Field

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single scatter event: no effect

- Larmor period certainly much larger than atomic distances
- muon unmagnetized as far as single scatter event goes

multiple scatter: weak effect on angles

- uniform B_Z field, go in Larmor frame, angles looks like B=0 case
- varying field, only affects distribution because scattered particles see different fields.
- For rapidly varying B, may enhance angles by defocussing;
- but typically place absorber symmetric about B=0 plane.

Magnetic field can modify shifts in position, but these are currently neglected anyway.

Path length changes are small (affects energy loss calculation?)

Large Step Size Errors

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Take B=0, neglect ΔP (energy loss)

$\delta P_X^2 \equiv P_Z^2\, \theta_{RMS}^2 = P_Z^2\, S\, \Delta z$, expected change in P_X^2

integrate to get other moments:

$\delta(xP_X) = P_X^2\, \Delta z / P_Z + P_Z\, S\, (\Delta z)^2 / 2$

$\delta x^2 = 2\, x\, P_X\, \Delta z / P_Z + P_X^2\, (\Delta z)^2 / P_Z^2 + S\, (\Delta z)^3 / 3$

can mimic effect with 2 displacements, one corr. with scatter, one uncorr.

Extra Emittance: dominant term is the usual $\beta_\perp\, P_Z\, S\, \Delta z / (2mc)$

new terms compete with $\beta_\perp$: order $S\, (\Delta z)^3$

$-\alpha_\perp\, \Delta z + (1+\alpha_\perp^2)\, (\Delta z)^2 / (3\, \beta_\perp) + \theta_{RMS}^2\, (\Delta z)^2 / (12\, \langle x^2 \rangle)$

typically, $\alpha_\perp < 1$, so require $\Delta z \ll \beta_\perp$

Large Step Size Errors

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Necessary condition for applying scatter at end of step:

$\Delta z << \beta_\perp$

same as regular tracking accuracy, small phase advance per step
tracking is 2nd order, here we have linear errors:

- but rel. to emittance growth from scatter; small, stochastic terms
- usually dominated by fluctuations for small # particles

Usually $\alpha_\perp$ is small, and absorbers placed at minimum $\beta_\perp$ ($\alpha_\perp = 0$) -- don't expect any visible effect even when $\Delta z \sim \beta_\perp$

Muons have high energy loss for given scatter in material:

this makes them a candidate for ionization cooling

so, non-symmetric energy loss will be dominant source of error.

Summary

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Boris push has been adapted for spatial tracking

- rapid, requires only one field evaluation per step
- good conservation properties, space-symmetric

Moliere scatter has small corrections for cooling muons

- kinematics: intermediate mass, limits angle from scatter off electrons; also don't neglect m_μ/m_p
- care about emittance, not just angles: spatial displacement
- magnetic fields only affect spatial displacement, already small
- constraints on accuracy resemble tracking errors in vacuum; here, is linear times small quantity, vs. second order

These effects yield slight underestimate in emittance growth, except for spatial displacements; small effect, and can be accounted for.

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