MidTerm II

Chapters 7-8, 11-12
1. (5pts) Given: \( m \rightarrow \text{mass}, \quad \vec{v} \rightarrow \text{velocity}, \quad h \rightarrow \text{height}, \quad k \rightarrow \text{spring constant}, \quad d \rightarrow \text{length of extension or compression}, \quad I \rightarrow \text{moment of inertia}, \quad \omega \rightarrow \text{angular velocity}, \) rank \( E_{\text{TOT}} \) in order of decreasing magnitude:

(a) \( m = 2.0\, \text{kg}, \quad \vec{v} = 3.0\, \text{m/s}, \quad h = 5.0\, \text{m} \)
\[
E_{\text{TOT}} = K + U_g = \frac{1}{2}mv^2 + mgh = 9J + 98J = 107J
\]

(b) \( m = 1.5\, \text{kg}, \quad \vec{v} = 7.0\, \text{m/s}, \quad k = 0.7, \quad d = 0.5\, \text{m} \)
\[
E_{\text{TOT}} = K + U_{\text{spring}} = \frac{1}{2}mv^2 + \frac{1}{2}kd^2 = 36.75J + 0.09J = 36.84J
\]

(c) \( m = 3.5\, \text{kg}, \quad \vec{v} = 4.3\, \text{m/s}, \quad I = 4.5\, \text{kgm}^2, \quad \omega = 3.0\, \text{rad/s} \)
\[
E_{\text{TOT}} = K_{\text{trans}} + K_{\text{rotate}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 32.35J + 20.25J = 52.60J
\]

(d) \( I = 6.0\, \text{kgm}^2, \quad \omega = 4.0\, \text{rev/s}, \quad h = 5\, \text{m} \)
\[
E_{\text{TOT}} = K_{\text{rotate}} + U_g = \frac{1}{2}I\omega^2 + mgh = \frac{1}{2}I\omega^2 + \frac{3}{2}Igh = 48J + 441J = 489J
\]

(e) \( E_{\text{TOT},d} > E_{\text{TOT},a} > E_{\text{TOT},c} > E_{\text{TOT},b} \)

2. (5pts) Give short physical examples of cases in Problem 1 and e) \( m, \ \vec{v}, \ h, \ k, \ d: \)

(a) free falling body mass \( m \)

(b) mass \( m \) moving on a spring on a horizontal frictionless surface

(c) smoothly rolling body

(d) rotating spherical shell at height \( h, \) but not falling

(e) bungy jumper
3. (5pts) In the figure below, a skater slides down three adjacent and different slopes on frictionless ice. Each slope is separated by a vertical distance \( d \). Rank the slopes (greatest to least) according to:

![Image of slopes]

(a) the work done on the skater by the gravitational force during the descent on each slope

The work done on an object by a change in the potential energy is given by \( W = -\Delta U \). Here, the change in potential energy is \( \Delta U = \Delta U_g = mgd \). Since \( d \) is the same for each slope, the work done on the skater by the gravitational potential energy is the same, such that \( W_1 = W_2 = W_3 \).

(b) the change in the skater’s kinetic energy along each slope

Since there is no friction and/or drag in the problem, the system is a conservative system and therefore, energy is conserved. So for each slope, the change in potential energy, \( \Delta U_g \), has the same magnitude as the change in kinetic energy, \( K \). Hence \( K_1 = K_2 = K_3 \).

4. (5pts) In the overhead view of the figure below, five forces of equal magnitude act upon a square plate which rotates about point \( P \). Rank the torques generated by each force, greatest first.

Generally, \( \tau = \vec{r} \times \vec{F} \), such that \( |\tau| = |\vec{r}| |\vec{F}| \sin \phi \).

Choosing the sides of the square to have dimension \( d \), then:

(a) \( |\vec{\tau}_1| = \frac{d}{2} \cdot |\vec{F}| \sin (90^\circ + \phi_1), (\phi_1 > \phi_2) \)

(b) \( |\vec{\tau}_2| = \frac{d}{2} \cdot |\vec{F}| \sin (90^\circ + \phi_2) \)

(c) \( |\vec{\tau}_3| = \frac{d}{2} \cdot |\vec{F}| \sin 0 = 0 \)

(d) \( |\vec{\tau}_4| = \frac{\sqrt{5}d}{2} \cdot |\vec{F}| \sin (90^\circ + \phi_4), (\phi_4 > \phi_5) \)

(e) \( |\vec{\tau}_5| = \frac{\sqrt{5}d}{2} \cdot |\vec{F}| \sin (90^\circ + \phi_5) \)

Hence \( |\vec{\tau}_5| > |\vec{\tau}_4| > |\vec{\tau}_2| > |\vec{\tau}_1| > |\vec{\tau}_3| \)
5. (5pts) In the table below, fill in the empty boxes where there are question marks:

<table>
<thead>
<tr>
<th>LINEAR</th>
<th>ANGULAR</th>
<th>RELATIONSHIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
<td>Name</td>
<td>Variable</td>
</tr>
<tr>
<td>( \vec{r} )</td>
<td>position</td>
<td>( \theta )</td>
</tr>
<tr>
<td>( \vec{v} = d\vec{r}/dt )</td>
<td>velocity</td>
<td>( \omega = d\theta/dt )</td>
</tr>
<tr>
<td>( a = d\vec{v}/dt )</td>
<td>acceleration</td>
<td>( \alpha = d\omega/dt )</td>
</tr>
<tr>
<td>( a = d^2\vec{r}/dt^2 )</td>
<td></td>
<td>( \alpha = d^2\theta/dt^2 )</td>
</tr>
<tr>
<td>( m )</td>
<td>mass</td>
<td>( I )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>&amp;</td>
</tr>
<tr>
<td>( \vec{p} = m\vec{v} )</td>
<td>momentum</td>
<td>( L = I\omega )</td>
</tr>
<tr>
<td>( \vec{F} = m\vec{a} )</td>
<td>force</td>
<td>( \tau = I\alpha )</td>
</tr>
<tr>
<td>( K = \frac{1}{2}mv^2 )</td>
<td>kinetic energy</td>
<td>( K = \frac{1}{2}I\omega^2 )</td>
</tr>
</tbody>
</table>

6. (5pts) Given \( \vec{r} = -3.5\hat{i} + 4.6\hat{j} \) in \( m \), \( \vec{F}_1 = 9.2\hat{j} + 7.0\hat{i} \) and \( \vec{F}_2 = 6.7\hat{i} + 1.8\hat{k} \) in \( N \), and \( \vec{p}_3 = 0.5\hat{k} - 4.6\hat{j} \) and \( \vec{p}_4 = -6.6\hat{i} - 6.6\hat{j} \) in \( kg \cdot m/s \), where the subscripts of the force and momentum vectors are the same as the subscripts of the torque and angular momentum vectors. Compute the following about the origin:

(a) \( \vec{\tau}_1 = \vec{r} \times \vec{F}_1 = (-3.5\hat{i} + 4.6\hat{j})m \times (7.0\hat{i} + 9.2\hat{j})N \)
\( = [( -3.5)(9.2)(\hat{i} \times \hat{j}) + (4.6)(7.0)(\hat{j} \times \hat{i})]N \cdot m \)
\( = [-32.2\hat{k} + 32.2\hat{j}]N \cdot m = -64.4\hat{k}N \cdot m \)

(b) \( \vec{\tau}_2 = \vec{r} \times \vec{F}_2 = (-3.5\hat{i} + 4.6\hat{j})m \times (6.7\hat{i} + 1.8\hat{k})N \)
\( = [[ -3.5](1.8)(\hat{i} \times \hat{k}) + (4.6)(6.7)(\hat{j} \times \hat{i}) + (4.6)(1.8)(\hat{j} \times \hat{k})]N \cdot m \)
\( = (8.3\hat{i} + 6.3\hat{j} - 30.8\hat{k})N \cdot m \)

(c) \( \vec{L}_3 = \vec{r} \times \vec{p}_3 = (-3.5\hat{i} + 4.6\hat{j})m \times (-4.6\hat{j} + 0.5\hat{k})kg \cdot m/s \)
\( = [( -3.5)(-4.6)(\hat{i} \times \hat{j}) + (3.5)(0.5)(\hat{i} \times \hat{k}) + (4.6)(0.5)(\hat{j} \times \hat{k})]kg \cdot m^2/s \)
\( = (2.3\hat{i} + 1.8\hat{j} + 16.1\hat{k})kg \cdot m^2/s \)

(d) \( \vec{L}_4 = \vec{r} \times \vec{p}_4 = (-3.5\hat{i} + 4.6\hat{j})m \times (-6.6\hat{i} - 6.6\hat{j})kg \cdot m/s \)
\( = [( -3.5)(-6.6)(\hat{i} \times \hat{j}) + (4.6)(-6.6)(\hat{j} \times \hat{i})]kg \cdot m^2/s \)
\( = (23.1\hat{k} + 30.4\hat{k})kg \cdot m^2/s = 53.5\hat{k} kg \cdot m^2/s \)

\( \vec{L}_2(t) = \int dt \cdot \vec{\tau}_2 = \int dt \vec{\tau}_2 t = (8.3\hat{i} + 6.3\hat{j} - 30.8\hat{k})t \ kg \cdot m^2/s \)

where: \( \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \),
\( \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{i} = \hat{j}, \hat{k} \times \hat{j} = -\hat{i}, \hat{k} \times \hat{i} = -\hat{j}, \hat{i} \times \hat{k} = -\hat{j} \)
7. (10pts) In the figure below, a frictionless roller coaster of mass \( m \) tops the first hill \( A \) with speed \( v_0 \). How much work does the gravitational force do on the car from point \( A \) to: (a) \( B \), to (b) \( C \), to (c) \( D \)? If the gravitational potential energy of the earth-coaster system is taken to be zero at point \( D \), what is its value when the coaster is at (d) point \( C \), and at point (e) \( B \)? (f) If the mass \( m \) of the car were doubled, would the change in the gravitational potential energy of the system between points \( B \) and \( C \) increase, decrease, or remain the same? (g) In terms of \( m \) and \( v_0 \), what is the total energy of the system if \( U_g = 0 \) at \( D \)?

\[ 
\text{In this problem } U = U_g, \text{ such that here } W = -\Delta U_g = -mg \Delta y 
\]

(a) Since \( \Delta y = 0 \) between \( A \) and \( B \), \( W_{AB} = 0 \)

(b) Here, \( \Delta y = h/2 \), such that \( W_{AC} = -\frac{1}{2}mgh \)

(c) Here, \( \Delta y = h \), such that \( W_{AD} = -mgh \)

(d) \( U_g(C) = \frac{1}{2}mgh \)

(e) \( U_g(B) = mgh \)

(f) For \( m \to 2m \), \( U_g(B) = 2mgh \) and \( U_g(C) = mgh \), so \( \Delta U \) would increase

(g) \( E_{TOT} = K + U_g = \frac{1}{2}mv_0^2 + mgh \)
8. (10pts) Two 70kg acrobats are entertaining a crowd with a ladder that pivots at the ground at point \( P \), but that they do not lean against a wall, see figure below. One acrobat holds the 5.5m ladder at a height of 1.5m and can apply a force, \( F_2 \), of up to twice his weight at an angle of 45° with respect to the ladder, \( \theta = 45^\circ \), while the other acrobat climbs to the top of the ladder. Use the approximation that the climbing acrobat has a center of mass along the axis of the ladder at the very top. What is the maximum angle at which the \( F_2 \) acrobat can lean the ladder before his colleague plummets to the ground; i.e. find \( \phi \)?

In order that the acrobat at the top of the ladder not fall, the bottom acrobat needs to apply enough force to balance the torque created by the upper acrobat, such that:

\[ |\vec{\tau}_2| = |\vec{\tau}_1|, \text{ in opposite directions.} \]

Using the Right–Hand–Rule: \( \vec{\tau}_1 \) is into plane of page \( \vec{\tau}_2 \) is out of plane of page

So:

\[ |\vec{\tau}_2| = |\vec{r}_2||\vec{F}_2| \sin \theta = |\vec{r}_1||\vec{F}_1| \sin \phi \]

where:

\[ |\vec{F}_1| = mg \]
\[ |\vec{F}_2| = 2 \cdot mg \]

Solving for \( \phi \):

\[ \phi = \sin^{-1} \left( \frac{|\vec{r}_2||\vec{F}_2| \sin \theta}{|\vec{r}_1||\vec{F}_1|} \right) \]
\[ = \sin^{-1} \left( \frac{|\vec{r}_2| \cdot 2mg \cdot \sin 45^\circ}{|\vec{r}_1| \cdot mg} \right) \]
\[ = \sin^{-1} \left( 2 |\vec{r}_2| \sin 45^\circ / |\vec{r}_1| \right) \]
\[ = \sin^{-1} \left( 2 (1.5m) \sin 45^\circ / 5.5m \right) \]

Finally:

\[ \phi \leq 0.4 \text{rad} = 22.9^\circ \]
9. (10pts) An automobile has a total mass of 1,700 kg. It accelerates from rest to 40 km/h in 10 s. Assume that each wheel is a uniform disk of mass 32 kg. Find, for the end of the 10 s interval, (a) the rotational kinetic energy of each wheel about its axle, (b) the total kinetic energy of each wheel, and (c) the total kinetic energy of the automobile. (d) Why is the radius of the wheel not necessary?

\[ v = 40 \text{km/h} = 40 \times 10^3 \text{m/h} \cdot (1 \text{h}/3600 \text{s}) = 11.1 \text{m/s} \]

Assuming that the wheels are uniform disks, \( I = \frac{1}{2} m R^2 \)
And using the relationship \( v = \omega r \):

(a) \( K_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{1}{2} \left( \frac{1}{2} m R^2 \right) (v/R)^2 = \frac{1}{4} m v^2 = \frac{1}{4} (32 \text{kg})(11.1 \text{m/s})^2 = 985.7 \text{J} \)

(b) \( K_{\text{wheel}} = K_{\text{rot}} + K_{\text{trans}} = \frac{1}{4} m v^2 + \frac{1}{2} m v^2 = \frac{3}{4} m v^2 \\
= \frac{3}{4} (32 \text{kg})(11.1 \text{m/s})^2 = 2,957.0 \text{J} \)

(c) \( K_{\text{auto}} = 4 \cdot K_{\text{rot}} + K_{\text{body}} = 4 \left( \frac{1}{4} m_{\text{wheel}} v^2 \right) + \frac{1}{2} m_{\text{body}} v^2 = m_{\text{wheel}} v^2 + \frac{1}{2} m_{\text{body}} v^2 \\
= (32 \text{kg})(11.1 \text{m/s})^2 + \frac{1}{2}(1,700 \text{kg})(11.1 \text{m/s})^2 \\
= 3,942.7 \text{J} + 104,728.5 \text{J} \\
= 108,671.2 \text{J} \)

(d) In part (b), the radii cancel